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Aharonov–Bohm scattering on two antiparallel flux lines of the same magnitude—without return flux

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Abstract. The problem of Aharonov–Bohm (AB) scattering on two parallel flux lines of arbitrary magnitude is solved exactly; the expression for scattering cross-section in the region of geometric shadow is derived. It is shown that in the particular case of two antiparallel lines of the same magnetic flux, though the return flux does not exist, the AB scattering still exists.

1. Introduction

We have solved exactly the AB scattering [1] on two parallel flux lines and flux tubes of the same magnitude [2–4]. Now we shall further generalize our method to solve exactly the AB scattering on two flux lines of arbitrary magnitude, the fluxes are $\beta_1\Phi$ and $\beta_2\Phi$, respectively. In the particular case of two antiparallel lines of the same magnitude ($\beta_1 = -\beta_2 = 1$), the return flux [5] does not exist, but we shall show that AB scattering still occurs in this case by the calculation of scattering cross-section. This result is very important in understanding the cause of AB scattering.

2. Vector potential

Let OXY be the coordinate plane perpendicular to two flux lines, and the coordinates of the two flux lines be $(a, 0)$ and $(-a, 0)$, respectively. We choose two polar coordinates (ρ_1, ϕ_1) and (ρ_2, ϕ_2) with these two points as poles. In the Coulomb gauge, the vector potential is

$$A = \frac{\Phi}{2\pi}\left(\frac{\beta_1 e_{\phi_1}}{\rho_1} + \frac{\beta_2 e_{\phi_2}}{\rho_2}\right) \tag{1}$$

where e_{ϕ_1} and e_{ϕ_2} are the unit vectors in the transverse direction of the two polar coordinates. In terms of elliptical coordinates (μ, θ)

$$x = a\cosh\mu\cos\theta \qquad y = a\sinh\mu\sin\theta \tag{2}$$

(1) becomes

$$\mathbf{A} = \frac{\Phi}{2\pi h}\left[-\frac{(\beta_1-\beta_2)\cosh\mu\sin\theta + (\beta_1+\beta_2)\sin\theta\cos\theta}{\cosh^2\mu - \cos^2\theta}e_\mu + \frac{(\beta_1-\beta_2)\sinh\mu\cos\theta + (\beta_1+\beta_2)\sinh\mu\cosh\mu}{\cosh^2\mu-\cos^2\theta}e_\theta\right]$$

$$(h \equiv [\cosh^2\mu - \cos^2\theta]^{1/2}). \tag{3}$$

Now we simplify the form of vector potential by a gauge function Λ. Let the coefficient of e_μ of the new vector potential equal zero, we obtain

$$\Lambda = \frac{\Phi}{2\pi}\left\{(\beta_1-\beta_2)\tan^{-1}\frac{\sinh\mu}{\sin\theta} + \frac{(\beta_1+\beta_2)}{2}\left[\sin^{-1}\frac{\cosh\mu\cos\theta-1}{\cosh\mu-\cos\theta} + \sin^{-1}\frac{\cosh\mu\cos\theta+1}{\cosh\mu+\cos\theta}\right] + 2g(\theta)\right\} \tag{4}$$

where $g(\theta)$ is a certain function of θ. The new vector potential is

$$\mathbf{A}' = \frac{\Phi}{\pi h}g'(\theta)\mathbf{e}_\theta \qquad g'(\theta) \equiv \frac{\mathrm{d}g(\theta)}{\mathrm{d}\theta}. \tag{5}$$

Equation (5) must satisfy the physical demand that

$$\oint_{C_1}\mathbf{A}'\cdot\mathrm{d}\gamma = \beta_1\Phi \qquad \oint_{C_2}\mathbf{A}'\cdot\mathrm{d}\gamma = \beta_2\Phi \tag{6}$$

where C_1 and C_2 are two closed paths around each flux. From (6) we get

$$g(\theta) = -\left[\frac{(\beta_1-\beta_2)}{2}\frac{\pi}{2}(1-\sin\theta) + \frac{(\beta_1+\beta_2)}{2}\left(\frac{\pi}{2}-\theta\right)\right]. \tag{7}$$

Substituting (7) into (5) we obtain

$$\mathbf{A}' = \frac{\Phi}{\pi h}\left(\frac{(\beta_1-\beta_2)}{2}\frac{\pi}{2}\cos\theta + \frac{(\beta_1+\beta_2)}{2}\right]\mathbf{e}_\theta. \tag{8}$$

3. Schrödinger equation

The Schrödinger equation is

$$\left(\nabla - \frac{ie}{\hbar c}\mathbf{A}'\right)^2\psi' = -k^2\psi' \tag{9}$$

where $k \equiv (2mE/\hbar^2)^{1/2}$ is the wavenumber. By writing

$$\psi' = M(\mu)\Theta(\theta) = M(\nu)Q(\theta)\exp\left\{-i\alpha\left[(\beta_1-\beta_2)\frac{\pi}{2}\sin\theta + (\beta_1+\beta_2)\theta\right]\right\} \tag{10}$$

we find

$$\frac{\mathrm{d}^2M}{\mathrm{d}\nu^2} + (\lambda - 2q\cos 2\nu)M = 0$$

$$\frac{\mathrm{d}^2Q}{\mathrm{d}\theta^2} + (\lambda - 2q\cos 2\theta)Q = 0 \tag{11}$$

where $\alpha \equiv -e\Phi/2\pi\hbar c$ is the quantum number of flux, $\nu = i\mu$, $q = a^2k^2/4$, $\lambda + 2q$ is the constant introduced in the seperation of variables. Equations (11) are recognized as the Mathieu equations. Using the general solution of (11) and the relations between wavefunction ψ (corresponding to **A**) and ψ' (corresponding to **A**′) we get

$$\psi = \exp\left[i\alpha(\beta_1 - \beta_2)\left(\tan^{-1}\frac{\sinh\mu}{\sin\theta} - \frac{\pi}{2}\right) + i\alpha\frac{\beta_1+\beta_2}{2}\left(\sin^{-1}\frac{\cosh\mu\cos\theta - 1}{\cosh\mu - \cos\theta} + \sin^{-1}\frac{\cosh\mu\cos\theta + 1}{\cosh\mu + \cos\theta} - \pi\right)\right]$$
$$\times \sum_{n=0}^{\infty} \{(A_n Ce_{2n}(\mu, q) + \bar{A}_n Fey_{2n}(\mu, q)]ce_{2n}(\theta, q)$$
$$+ [B_n Ce_{2n+1}(\mu, q) + \bar{B}_n Fey_{2n+1}(\mu, q)]ce_{2n+1}(\theta, q)$$
$$+ [C_n Se_{2n+1}(\mu, q) + \bar{C}_n Gey_{2n+1}(\mu, q)]se_{2n+1}(\theta, q)$$
$$+ [D_n Se_{2n+2}(\mu, q) + \bar{D}_n Gey_{2n+2}(\mu, q)]se_{2n+2}(\theta, q)\}. \tag{12}$$

In order to satisfy the initial condition that the current density

$$\mathbf{j} = \frac{\hbar(\psi^*\nabla\psi - \psi\nabla\psi^*)}{2im} - \frac{e}{mc}\mathbf{A}\psi^*\psi \tag{13}$$

should be constant and in the x direction, the incident wave must be chosen as

$$\psi_{\text{inc}} = \exp\{-2i\alpha[g(\theta) + g_0] - ikx\}. \tag{14}$$

For simplicity we choose the arbitrary constant g_0 as

$$g_0 = \left(\frac{\beta_1 - \beta_2}{2} + \frac{\beta_1 + \beta_2}{2}\right)\frac{\pi}{2} = \frac{\pi}{2}\beta_1 \tag{15}$$

hence (14) becomes

$$\psi_{\text{inc}} = \exp\left\{ -i(\beta_1 + \beta_2)\alpha\theta - i(\beta_1 - \beta_2)\alpha\frac{\pi}{2}\sin\theta + ik\rho\sin(\theta + \tau)\right\} \tag{16}$$

where τ is the angle between the wavevector $\boldsymbol{k}$ of the incident wave and the y axis.

Considering the expression (16) of the incident wave, we rewrite (12) as

$$\psi=\exp\left\{-\mathrm{i}(\beta_1-\beta_2)\alpha\frac{\pi}{2}\sin\theta\right\}\cdot\left\{\sum_{m=0}^{\infty}\sum_{l}[C^c_{mlq}Ce_l(\mu,q)\right.$$
$$+\bar{C}^c_{mlq}Fey_l(\mu,q)+S^c_{mlq}Se_l(\mu,q)+\bar{S}^c_{mlq}Gey_l(\mu,q)]ce_m(\theta,q)$$
$$+\sum_{m=1}^{\infty}\sum_{l}[C^s_{mlq}Ce_l(\mu,q)+\bar{C}^s_{mlq}Fey_l(\mu,q)+S^s_{mlq}Se_l(\mu,q)$$
$$+\bar{S}^s_{mlq}Gey_l(\mu,q)]se_m(\theta,q)$$
$$=\exp\left\{-\mathrm{i}(\beta_1-\beta_2)\alpha\frac{\pi}{2}\sin\theta\right\}\cdot\left\{\sum_{m=0}^{\infty}\sum_{l}[(C^c_{ml}+c^c_{ml}q+O(q^2))Ce_l(\mu,q)\right.$$
$$+(\bar{C}^c_{ml}+\bar{c}^c_{ml}q+O(q^2))Fey_l(\mu,q)+(S^c_{ml}+s^c_{ml}q+O(q^2))Se_l(\mu,q)$$
$$+(\bar{S}^c_{ml}+\bar{s}^c_{ml}q+O(q^2))Gey_l(\mu,q)]ce_m(\theta,q)$$
$$+\sum_{m=1}^{\infty}\sum_{l}[(C^s_{ml}+c^s_{ml}q+O(q^2))Ce_l(\mu,q)$$
$$+(\bar{C}^s_{ml}+\bar{c}^s_{ml}q+O(q^2))Fey_l(\mu,q)$$
$$+(S^s_{ml}+s^s_{ml}q+O(q^2))Se_l(\mu,q)$$
$$\left.+(\bar{S}^s_{ml}+\bar{s}^s_{ml}q+O(q^2))Gey_l(\mu,q)]se_m(\theta,q)\right\} \tag{17}$$

where the coefficients C^c_{ml}, $\bar{C}^c_{ml}$, S^c_{ml}, $\bar{S}^c_{ml}$, ... are functions of α only. Using the same method as [1], we can find these coefficients under the conditions $\mu\to\infty$ and $q\to 0$. Let

$$\Theta=\exp\left\{-\mathrm{i}\alpha(\beta_1-\beta_2)\frac{\pi}{2}\sin\phi+\mathrm{i}m\phi\right\} \tag{18}$$

we obtain

$$\lambda=[m+(\beta_1+\beta_2)\alpha]^2 \tag{19}$$

and hence

$$R''+\frac{1}{\rho}R'+\left\{k^2-\frac{[m+(\beta_1+\beta_2)\alpha]^2}{\rho^2}\right\}R=0 \tag{20}$$

where $R(\rho)=M(\mu)$, (ρ,ϕ) are polar coordinates with the origin of the rectangular coordinates as pole. Through quite a tedious calculation as that given in [1], we find

$$C^c_{00}=\frac{1}{2p'_0}(\mathrm{e}^{-2\mathrm{i}\delta}+1)\qquad \bar{C}^c_{00}=\frac{\mathrm{i}}{2p'_0}(\mathrm{e}^{-2\mathrm{i}\delta}-1),\left(\delta=\frac{1}{2}(\beta_1+\beta_2)\pi\alpha\right)$$

$$C^c_{ml}=\begin{cases}\dfrac{2}{p'_l}\cos(2n\tau-\delta)\cos\delta & l=m=2n\neq 0\\ \dfrac{2\mathrm{i}}{p'_l}\sin[(2n+1)\tau-\delta]\cos\delta & l=m=2n+1\\ 0 & l\neq m\end{cases}$$

$$\tilde{C}^c_{ml}=\begin{cases}\dfrac{2\mathrm{i}}{p'_l}\sin(2n\tau-\delta)\sin\delta & l=m=2n\neq 0\\ \dfrac{2}{p'_l}\cos[(2n+1)\tau-\delta]\sin\delta & l=m=2n+1\\ 0 & l\neq m\end{cases}$$

$$S^s_{ml}=\begin{cases}\dfrac{2\mathrm{i}}{s'_l}\cos[(2n+1)\tau-\delta]\cos\delta & l=m=2n+1\\ \dfrac{-2}{s'_l}\sin[(2n+2)\tau-\delta]\cos\delta & l=m=2n+2\\ 0 & l\neq m\end{cases}$$

$$\tilde{S}^s_{ml}=\begin{cases}\dfrac{-2}{s'_l}\sin[(2n+1)\tau-\delta]\sin\delta & l=m=2n+1\\ \dfrac{2\mathrm{i}}{s'_l}\cos[(2n+1)\tau-\delta]\sin\delta & l=m=2n+2\\ 0 & l\neq m\end{cases}$$

$$S^c_{ml}=\tilde{S}^c_{ml}=C^s_{ml}=\tilde{C}^s_{ml}=0 \tag{21}$$

the constant multipliers p'_l and s'_l are given in [6, pp. 368–369].

We comment here on criticism [7] of this method of calculation. First, (8) is calculated from two magnetic flux lines, it is singular only at two foci F_1 and F_2, not at other points on the line $\overline{F_1F_2}$, i.e. there is no singularity for $\mathbf{A}'$ on the line $\overline{F_1F_2}$ except at the two ends. Second, the singularity of elliptical coordinates consists only in the multivaluedness of θ on the line $\overline{F_1F_2}$, but this singularity can be removed by recognizing $\overline{F_1F_2}$ as a branch line. Thirdly, there is no singularity for our solution (12) on the line $\overline{F_1F_2}$; in the appendix we show explicitly that this solution obeys $\psi(\mu=0,\theta)=\psi(\mu=0,-\theta)$. Finally, we do not use any nearby boundary conditions to determine the coefficients in (17), we use only the faraway boundary conditions. Therefore our results (17) and (21) are related to the scattering of electrons by two magnetic flux lines, not by a magnetic flux which is spread continuously along the line $\overline{F_1F_2}$ [7], and our method of calculation is correct.

4. Scattering cross-section

In the asymptotic region $\phi=\theta$

$$\psi=\exp\left\{-\mathrm{i}(\beta_1+\beta_2)\alpha\theta-\mathrm{i}(\beta_1-\beta_2)\alpha\frac{\pi}{2}\sin\theta+\mathrm{i}k\rho\sin(\theta+\tau)\right\}+f(\theta)\frac{\mathrm{e}^{\mathrm{i}k\rho}}{\sqrt{k\rho}}. \tag{22}$$

Combining (17), (21) and (22), taking advantage of the orthogonality relations of Mathieu functions, we can find the expression for $f(\theta)$ as an expansion for the Mathieu functions. We find that the results are easily obtained from [1] by changing 2α to $(\beta_1+\beta_2)\alpha$ and by increasing a phase factor $\exp\{-i(\beta_1-\beta_2)\alpha(\pi/2)\sin\theta\}$.

Now we derive an expression for $f(\theta)$ in the region of the geometric shadow of the strings where q is very large but not yet infinite. When $qa\gg 1$, and $-\pi/2<\theta<\pi/2$ we have

$$
\begin{aligned}
f(\theta)=\exp\left\{-i(\beta_1-\beta_2)\alpha\frac{\pi}{2}\sin\theta\right\}\cdot\Bigg[&\tfrac{1}{2}(H_0^+ + h_0^+ q + O(q^2))ce_0(\theta,q)\\
&+\sum_{n=1}^{\infty}(H_{1n}^+ + h_{1n}^+ q + O(q^2))ce_{2n}(\theta,q)\\
&+\sum_{n=0}^{\infty}(H_{2n}^+ + h_{2n}^+ q + O(q^2))ce_{2n+1}(\theta,q)\\
&+\sum_{n=0}^{\infty}(H_{3n}^+ + h_{3n}^+ q + O(q^2))se_{2n+1}(\theta,q)\\
&+\sum_{n=0}^{\infty}(H_{4n}^+ + h_{4n}^+ q + O(q^2))se_{2n+2}(\theta,q)\Bigg]
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
&H_0^+ = \sqrt{\frac{2}{\pi}}e^{-(i\pi/4)-2i\delta} \qquad H_{1n}^+ = (-1)^n\sqrt{\frac{2}{\pi}}e^{-i\pi/4}\cos(2n\tau-2\delta)\\
&H_{2n}^+ = i(-1)^n\sqrt{\frac{2}{\pi}}e^{i\pi/4}\sin[(2n+1)\tau-2\delta]\\
&H_{3n}^+ = i(-1)^n\sqrt{\frac{2}{\pi}}e^{-i\pi/4}\cos[(2n+1)\tau-2\delta]\\
&H_{4n}^+ = (-1)^n\sqrt{\frac{2}{\pi}}e^{-i\pi/4}\sin[(2n+2)\tau-2\delta].
\end{aligned}
\tag{24}
$$

When $\alpha\to 0$, we must have $f(\theta)=0$, by the orthogonality of circular functions, equation (59) and the corrected version of equation (60) of [1] ((E2) of [2] should change to $-e^{-2i\delta}\cos(2\theta)$ in this case), we get

$$h_0^+ = \frac{e^{-i\pi/4}}{\sqrt{2\pi}}\left(\frac{\pi}{2}+\tau+\sin\tau\right) \qquad h_{1n}^+ = -\frac{e^{-i\pi/4}}{\sqrt{2\pi}}\frac{\sin\tau}{4n^2-1}$$
$$h_{2n}^+ = -\frac{e^{-i\pi/4}}{\sqrt{2\pi}}\frac{\sin\tau}{(2n+1)^2-1} \tag{25}$$
$$h_{3n}^+ = \frac{-e^{-i\pi/4}}{\sqrt{2\pi}}\frac{(2n+1)\cos\tau}{(2n+1)^2-1} \qquad h_{4n}^+ = -\frac{e^{-i\pi/4}}{\sqrt{2\pi}}\frac{(2n+2)\cos\tau}{(2n+2)^2-1}.$$

Substituting (24), (25) into (23) and using the approximate formula [6]

$$\begin{matrix} ce_m \\ se_m \end{matrix}(\theta, q) \approx \begin{matrix} p'_m \\ s'_{m+1} \end{matrix} \frac{2^{m-1/2}}{(\pi q^{1/2})^{1/2}\cos^{m+1}\theta}$$
$$\times\left\{e^{2q^{1/2}\sin\theta}\left[\cos\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right]^{2m+1} \pm e^{-2q^{1/2}\sin\theta}\left[\sin\left(\frac{\theta}{2}+\frac{\pi}{4}\right)\right]^{2m+1}\right\} \tag{26}$$

in the case $\tau = -\pi/2$ and $\theta \approx 0$, we get

$$f = -\frac{\sqrt{2}\,e^{-i\pi/4}}{2\pi q^{1/4}}\cos 2\delta\left\{p'_0 + 2\sum_{n=0}^{\infty}(p'_{2n+1}-p'_{2n}) + \frac{q}{\cos 2\delta}\sum_{n=0}^{\infty}\left(\frac{p'_{2n+1}}{(2n+1)^2-1} - \frac{p'_{2n}}{4n^2-1}\right)\right.$$
$$\left. + ip'_0\tan 2\delta\right\} \tag{27}$$

hence

$$\sigma = |f|^2 = \frac{\cos^2 2\delta}{2\pi^2 q^{1/2}}\left\{\left[p'_0 + 2\sum_{n=0}^{\infty}(p'_{2n+1}-p'_{2n}) + \frac{q}{\cos 2\delta}\sum_{n=0}^{\infty}\left(\frac{p'_{2n+1}}{(2n+1)^2-1} - \frac{p'_{2n}}{4n^2-1}\right)\right]^2\right.$$
$$\left. + p_0'^{2}\tan^2 2\delta\right\}. \tag{28}$$

In the particular case of two antiparallel lines of flux, $\beta_1 = -\beta_2 = 1$, (28) reduces to

$$\sigma = \frac{1}{2\pi^2 q^{1/2}}\left[p'_0 + 2\sum_{n=0}^{\infty}(p'_{2n+1}-p'_{2n}) + q\sum_{n=0}^{\infty}\left(\frac{p'_{2n+1}}{(2n+1)^2-1} - \frac{p'_{2n}}{4n^2-1}\right)\right]^2. \tag{29}$$

Obviously, the scattering cross-section does not equal zero in this case. Hence we obtain the important conclusion: even in the case that the return flux does not exist, AB scattering still exists.

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Appendix

Here we give the proof of the result:

$$\psi(\mu=0,\theta)=\psi(\mu=0,-\theta). \tag{A.1}$$

Using the following formulae [6, pp. 161–162]:

$$Se_{2n+1}(\mu,q)=\frac{se_{2n+1}(\pi/2,q)}{kB_1^{(2n+1)}}\tanh\mu\sum_{r=0}^{\infty}(-1)^r(2r-1)B_{2r+1}^{(2n+1)}J_{2n+1}(2k\cosh\mu)$$

(b_{2n+1})

$$Gey_{2n+1}(\mu,q)=\frac{se_{2n+1}(\pi/2,q)}{kB_1^{(2n+1)}}\tanh\mu\sum_{r=0}^{\infty}(-1)^r(2r+1)B_{2r+1}^{(2n+1)}Y_{2n+1}(2k\cosh\mu)$$

(b_{2n+1}) (A.2)

$$Se_{2n+1}(\mu,q)=\frac{-se'_{2n+2}(\pi/2,q)}{k^2B_2^{(2n+2)}}\tanh\mu\sum_{r=0}^{\infty}(-1)^r(2r+2)B_{2r+2}^{(2n+1)}J_{2n+2}(2k\cosh\mu)$$

(b_{2n+2})

$$Gey_{2n+2}(\mu,q)=\frac{-se'_{2n+2}(\pi/2,q)}{k^2B_2^{(2n+2)}}\tanh\mu\sum_{r=0}^{\infty}(-1)^r(2r+2)B_{2r+2}^{(2n+2)}Y_{2n+2}(2k\cosh\mu)$$

(b_{2n+2})

and the result $\tanh\mu|_{\mu=0}=0$, we get

$$\begin{aligned}&Se_m(0,q)=0\\&Gey_m(0,q)=0\end{aligned}\qquad m=2n+1,2n+2. \tag{A.3}$$

Using the following formulae [6, p. 21]:

$$ce_{2n}(\theta,q)=\sum_{r=0}^{\infty}A_{2r}^{(2n)}\cos 2r\theta \qquad (a_{2n})$$

$$ce_{2n+1}(\theta,q)=\sum_{r=0}^{\infty}A_{2r+1}^{(2n+1)}\cos(2r+1)\theta \qquad (a_{2n+1})$$ (A.4)

and the relation $\cos(-\phi)=\cos\phi$, we obtain

$$ce_m(-\theta,q)=ce_m(\theta,q)\qquad m=2n,2n+1. \tag{A.5}$$

Letting $\mu=0$ in (12), using $\sinh\mu|_{\mu=0}=0$ and $\cosh\mu|_{\mu=0}=1$, and applying (A.3), we get

$$\begin{aligned}\psi(\mu=0,\theta)=\exp(-\mathrm{i}\alpha\beta,\pi)\sum_{n=0}^{\infty}\{&[A_nCe_{2n}(0,q)+\bar{A}_nFey_{2n}(0,q)]ce_{2n}(\theta,q)\\&+[B_nCe_{2n+1}(0,q)+\bar{B}_nFey_{2n+1}(0,q)]ce_{2n+1}(\theta,q)\}.\end{aligned} \tag{A.6}$$

Letting $\theta\to-\theta$ in (A.6) and applying (A.5), we easily obtain the required result (A.1).

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